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Spectrum and Spectral Singularities of a Quadratic Pencil of a Schrödinger Operator with a General Boundary Condition

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In this article we investigate the spectrum and the spectral singularities of the Quadratic Pencil of Schrödinger Operator L generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\ell(y) = -y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y, \qquad x \in \mathbf{R}_+ = [0, \infty)$$

and the boundary condition

$$\int_0^\infty K(x)\, y(x)\, dx + \alpha y'(0) - \beta y(0) = 0,$$

where p, q, and K are complex valued functions, p is continuously differentiable on $\mathbf{R}_+$, $K \in L^2(\mathbf{R}_+)$, and $\alpha, \beta \in \mathbf{C}$, with $|\alpha| + |\beta| \neq 0$. Discussing the spectrum, we prove that L has a finite number of eigenvalues and spectral singularities with finite multiplicities, if the conditions

$$\lim_{x \to \infty} p(x) = 0, \qquad \sup_{x \in \mathbf{R}_+} \{e^{\varepsilon \sqrt{x}}[\,|q(x)| + |p'(x)| + |K(x)|\,]\} < \infty, \quad \varepsilon > 0.$$

Later we investigate the properties of the principal functions corresponding to the spectral singularities. Moreover, some results about the spectrum of L are applied to non-selfadjoint Sturm–Liouville and Klein–Gordon s-wave operators.

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1. INTRODUCTION

Let L_0 denote the operator generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\ell_0(y) = -y'' + q(x)\,y, \qquad x \in \mathbf{R}_+ = [0, \infty), \tag{1.1}$$

and the boundary condition $y'(0) - hy(0) = 0$, where q is a complex valued function, and $h \in \mathbf{C}$. The study of the spectral analysis of L_0 was begun by Naimark [22] in 1954. In this article, the spectrum of L_0 was investigated and shown that it is composed of eigenvalues, a continuous spectrum, and spectral singularities. Spectral singularities are poles of the resolvent's kernel which are imbedded in the continuous spectrum and are not eigenvalues. If

$$\int_0^\infty e^{\varepsilon x} |q(x)|, \qquad dx < \infty$$

for some $\varepsilon > 0$, then L_0 has a finite number of eigenvalues and spectral singularities with finite multiplicities. Moreover, the spectral expansion was derived in some particular cases.

The result of Naimark was extended to a differential operator on the whole real axis by Kemp [11].

One very important step in the spectral analysis of L_0 was taken by Pavlov [24]. In his article, he studied the dependence of the structure of spectral singularities of L_0 on the behavior of the potential function at infinity. Using the generalized spectral function of L_0 (in the sense of Marchenko [21]) and the analytical properties of Weyl–Titchmarsh function [27], he obtained the spectral expansion in terms of the principal functions of L_0. We should note that Pavlov's paper [24] is the first article in which the spectral expansion of L_0 is derived, taking into account the spectral singularities as well, but the effect of the spectral singularities in the spectral expansion was not discussed.

Lyance showed that the spectral singularities play an important role in the spectral analysis of L_0 [17]. He also investigated the effect of spectral singularities in the spectral expansion.

Spectral analysis of the non-selfadjoint operator L_1 generated in $L^2(\mathbf{R}_+)$ by (1.1) and the boundary condition

$$\int_0^\infty K(x)\, y(x)\, dx + \alpha y'(0) - \beta y(0) = 0,$$

in which $K \in L^2(\mathbf{R}_+)$ is a complex valued function, and $\alpha, \beta \in \mathbf{C}$, was investigated in detail by Krall [12–16]. In [12] he obtained the adjoint L_1^*

of the operator L_1. Note that L_1^* deserves a special interest, since it is not a purely differential operator. The eigenfunction expansion of L_1 and L_1^* and the Weyl theory of L_1 were investigated by Krall in [13] and [14], respectively.

Some problems of spectral theory of differential and some others types of operators with spectral singularities were studied by others in [4–6], [9, 18, 19, 23, 25, 26].

Let us consider a differential expression of the form

$$\ell(y) = -y'' + [q(x) + 2\lambda p(x) - \lambda^2]y, \qquad x \in \mathbf{R}_+,$$

where p and q are complex valued functions, p is continuously differentiable on $\mathbf{R}_+$ and bounded.

We denote by D_0 those functions f defined on $\mathbf{R}_+$ and satisfying

(i) $f \in L^2(\mathbf{R}_+)$,

(ii) f' exists and absolutely continuous on every finite subinterval of $\mathbf{R}_+$,

(iii) $\ell(f) \in L^2(\mathbf{R}_+)$.

Let K be an arbitrary complex-valued function in $L^2(\mathbf{R}_+)$, $\alpha, \beta \in \mathbf{C}$ and $|\alpha| + |\beta| \neq 0$. We denote by D those functions f satisfying

(i) $f \in D_0$,

(ii) $\int_0^\infty K(x) f(x)\, dx + \alpha f'(0) - \beta f(0) = 0$.

It is clear that, D is dense in $L^2(\mathbf{R}_+)$ ([13]).

We define the operator L by $Lf = \ell(f)$ for all f in D.

The operators L_0 and L_1 are particular cases of L. If $K \equiv 0$, $\alpha = 0$ and $\beta = 1$ then L will reduce to the operator L_2 defined by

$$\ell(y) = -y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y, \qquad x \in \mathbf{R}_+,$$

and the boundary condition $y(0) = 0$. Some problems of the spectral theory of L_2 were studied by Jaulent–Jean [10], Maksudov [18], and Maksudov–Guseinov [20].

Let L_3 denote the operator generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\tilde{\ell}(y) = -y'' - [\lambda - p(x)]^2\, y, \qquad x \in \mathbf{R}_+,$$

and the boundary condition $y(0) = 0$. L_3 is a particular case of L ($K(x) \equiv 0$, $q(x) = -p^2(x)$, $\alpha = 0$, $\beta = 1$). Note that in relativistic quantum mechanics the equation

$$y'' + [\lambda - p(x)]^2\, y = 0, \qquad x \in \mathbf{R}_+$$

is called the Klein–Gordon s-wave equation for a particle of zero mass with static potential p ([7]). The eigenvalues of L_3 were investigated by Degasperis in the case that p is real, analytic and vanishes rapidly for $x \to \infty$, [3].

In this paper, we discuss the discrete spectrum of L and prove that this operator has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if

$$\lim_{x \to \infty} p(x) = 0, \qquad \sup_{x \in \mathbf{R}_+} \{ e^{\varepsilon \sqrt{x}} [\, |q(x)| + |p'(x)| + |K(x)| \,] \} < \infty, \quad \varepsilon > 0.$$

Then we obtain the properties of the principal functions corresponding to the spectral singularities of L.

2. SOLUTIONS OF $\ell(y) = 0$.

Suppose the functions p and q satisfy

$$\int_0^\infty |p(x)|\, dx < \infty, \qquad \int_0^\infty x[\, |q(x)| + |p'(x)| \,]\, dx < \infty. \tag{2.1}$$

Under (2.1) the equation $\ell(y) = 0$ has the solutions

$$e^+(x, \lambda) = e^{iw(x) + i\lambda x} + \int_x^\infty A^+(x, t)\, e^{i\lambda t}\, dt \tag{2.2}$$

and

$$e^-(x, \lambda) = e^{-iw(x) - i\lambda x} + \int_x^\infty A^-(x, t)\, e^{-i\lambda t}\, dt \tag{2.3}$$

for $\lambda \in \overline{\mathbf{C}}_+ = \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda \geqslant 0\}$ and $\lambda \in \overline{\mathbf{C}}_- = \{\lambda : \lambda \in \mathbf{C}, \operatorname{Im} \lambda \leqslant 0\}$, respectively, where $w(x) = \int_x^\infty p(t)\, dt$, and the kernels $A^\pm(x, t)$ may be expressed in terms of p and q. $A^\pm(x, t)$ are continuously differentiable with respect to their arguments and

$$|A^\pm(x, t)| \leqslant C\xi\left(\frac{x+t}{2}\right) \exp\{\zeta(x)\}, \tag{2.4}$$

$$|A_x^\pm(x, t)|,\ |A_t^\pm(x, t)| \leqslant C\left[\, \xi^2\left(\frac{x+t}{2}\right) + \theta\left(\frac{x+t}{2}\right) \right], \tag{2.5}$$

where

$$\xi(x)=\int_x^\infty [|q(t)|+|p'(t)|]\,dt, \qquad \zeta(x)=\int_x^\infty [t\,|q(t)|+2\,|p(t)|]\,dt, \tag{2.6}$$

$$\theta(x)=\tfrac{1}{4}[|q(x)|+|p(x)|^2+|p'(x)|], \tag{2.7}$$

and $C>0$ is a constant. Therefore $e^+(x,\lambda)$ and $e^-(x,\lambda)$ are analytic with respect to λ in $\mathbf{C}_+=\{\lambda\colon \lambda\in\mathbf{C},\ \operatorname{Im}\lambda>0\}$ and $\mathbf{C}_-=\{\lambda\colon \lambda\in\mathbf{C},\ \operatorname{Im}\lambda<0\}$, respectively, and continuous up to the real axis. $e^\pm(x,\lambda)$ also satisfy

$$e^\pm(x,\lambda)=e^{\pm i[w(x)+\lambda x]}+O\left(\frac{e^{\mp x\,\operatorname{Im}\lambda}}{|\lambda|}\right), \qquad \lambda\in\overline{\mathbf{C}}_\pm, \quad |\lambda|\to\infty, \tag{2.8}$$

$$e_x^\pm(x,\lambda)=\pm i[\lambda-p(x)]\,e^{\pm i[w(x)+\lambda x]}+O(1), \qquad \lambda\in\overline{\mathbf{C}}_\pm, \quad |\lambda|\to\infty. \tag{2.9}$$

Let $\varphi^\pm(x,\lambda)$ denote the solutions of $\ell(y)=0$ subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\lambda x}\varphi^\pm(x,\lambda)=1, \qquad \lim_{x\to\infty} e^{\pm i\lambda x}\varphi_x^\pm(x,\lambda)=\mp i\lambda, \quad \lambda\in\overline{\mathbf{C}}_\pm. \tag{2.10}$$

Then

$$W[e^\pm(x,\lambda),\varphi^\pm(x,\lambda)]=\mp 2i\lambda, \qquad \lambda\in\overline{\mathbf{C}}_\pm, \tag{2.11}$$

and

$$W[e^+(x,\lambda),e^-(x,\lambda)]=-2i\lambda, \qquad \lambda\in\mathbf{R}=(-\infty,\infty), \tag{2.12}$$

where $W[f_1,f_2]$is the Wronskian of f_1 and f_2, (Jaulent–Jean [10]).

3. THE RESOLVENT AND THE CONTINUOUS SPECTRUM OF L

Let us define the following functions:

$$\begin{aligned} N^\pm(\lambda)&=\int_0^\infty K(x)\,e^\pm(x,\lambda)\,dx+\alpha e_x^\pm(0,\lambda)-\beta e^\pm(0,\lambda),\\ g^\pm(t,\lambda)&=\mp\frac{1}{2i\lambda}\left\{e^\pm(t,\lambda)\int_t^\infty K(x)\,\varphi^\pm(x,\lambda)\,dx\right.\\ &\quad+\varphi^\pm(t,\lambda)\int_0^t K(x)\,e^\pm(x,\lambda)\,dx\\ &\quad\left.+[\alpha e_x^\pm(0,\lambda)-\beta e^\pm(0,\lambda)]\,\varphi^\pm(t,\lambda)\right\}. \end{aligned} \tag{3.1}$$

Let

$$G(x, t; \lambda) = \begin{cases} G^{+}(x, t; \lambda), & \lambda \in \mathbf{C}_{+}, \\ G^{-}(x, t; \lambda), & \lambda \in \mathbf{C}_{-}, \end{cases} \tag{3.2}$$

be the Green's function of L (obtained by the standard techniques), where

$$G^{\pm}(x, t; \lambda) = G_1^{\pm}(x, t; \lambda) + G_2^{\pm}(x, t; \lambda), \tag{3.3}$$

and

$$G_1^{\pm}(x, t; \lambda) = \frac{e^{\pm}(x, \lambda)\, g^{\pm}(t, \lambda)}{N^{\pm}(\lambda)}, \tag{3.4}$$

$$G_2^{\pm}(x, t; \lambda) = \pm \begin{cases} \dfrac{e^{\pm}(t, \lambda)\, \varphi^{\pm}(x, \lambda)}{2i\lambda}, & 0 \leqslant t < x \\ \dfrac{e^{\pm}(x, \lambda)\, \varphi^{\pm}(t, \lambda)}{2i\lambda}, & x \leqslant t < \infty. \end{cases} \tag{3.5}$$

We will denote the continuous spectrum of L by $\sigma_c(L)$. Using (3.2)–(3.5) in a way similar to Theorem 4.4 in [13], we have the following

THEOREM 3.1. $\sigma_c(L) = (-\infty, \infty)$.

4. THE EIGENVALUES AND THE SPECTRAL SINGULARITIES OF L

We see from (2.11) and (2.12) that if

$$\tilde{N}^{\pm}(\lambda) = \int_0^{\infty} K(x)\, \varphi^{\pm}(x, \lambda)\, dx + \alpha \varphi_x^{\pm}(0, \lambda) - \beta \varphi^{\pm}(0, \lambda),$$

then the functions $\psi^{\pm}(x, \lambda)$ and $\psi(x, \lambda)$, defined by

$$\psi^{\pm}(x, \lambda) = \tilde{N}^{\pm}(\lambda)\, e^{\pm}(x, \lambda) - N^{\pm}(\lambda)\, \varphi^{\pm}(x, \lambda), \qquad \lambda \in \mathbf{C}_{\pm}, \tag{4.1}$$

$$\psi(x, \lambda) = N^{+}(\lambda)\, e^{-}(x, \lambda) - N^{-}(\lambda)\, e^{+}(x, \lambda), \qquad \lambda \in \mathbf{R}^* = \mathbf{R} \backslash \{0\}, \tag{4.2}$$

are the solutions of the boundary value problem

$$-y'' + [q(x) + 2\lambda p(x) - \lambda^2]\, y = 0, \qquad x \in \mathbf{R}_{+},$$

$$\int_0^{\infty} K(x)\, y(x)\, dx + \alpha y'(0) - \beta y(0) = 0.$$

Let us denote the eigenvalues and the spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From (2.8), (2.10), (3.4), (4.1), and (4.2) it follows that

$$\sigma_d(L) = \{\lambda : \lambda \in \mathbf{C}_+, N^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{C}_-, N^-(\lambda) = 0\}, \tag{4.3}$$

$$\sigma_{ss}(L) = \{\lambda : \lambda \in \mathbf{R}^*, N^+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbf{R}^*, N^-(\lambda) = 0\}, \tag{4.4}$$

$$\{\lambda : \lambda \in \mathbf{R}^*, N^+(\lambda) = 0\} \cap \{\lambda : \lambda \in \mathbf{R}^*, N^-(\lambda) = 0\} = \varnothing.$$

Definition 4.1. The multiplicity of a zero N^+ (or N^-) in $\overline{\mathbf{C}}_+$ (or $\overline{\mathbf{C}}_-$) is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of L.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of N^+ and N^- in $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. For the sake of simplicity we will consider only the zeros of N^+ in $\overline{\mathbf{C}}_+$. A similar procedure may also be employed for zeros of N^- in $\overline{\mathbf{C}}_-$. Define

$$M_1^+ = \{\lambda : \lambda \in \mathbf{C}_+, N^+(\lambda) = 0\}, \qquad M_2^+ = \{\lambda : \lambda \in \mathbf{R}, N^+(\lambda) = 0\}.$$

Lemma 4.2. *Let $K \in L^1(\mathbf{R}_+) \cap L^2(\mathbf{R}_+)$. Under the condition* (2.1) *we have*

(i) *The set M_1^+ is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*

(ii) *The set M_2^+ is compact.*

Proof. From (2.2) we obtain that N^+ is analytic in $\mathbf{C}_+$, continuous in $\overline{\mathbf{C}}_+$, and has the form

$$\begin{aligned} N^+(\lambda) = {} & i\lambda\alpha e^{iw(0)} - \alpha[ip(0)\, e^{iw(0)} + A^+(0,0)] \\ & - \beta e^{iw(0)} + \int_0^\infty f^+(t)\, e^{i\lambda t}\, dt, \end{aligned} \tag{4.5}$$

where

$$f^+(t) = K(t)\, e^{iw(t)} + \int_0^t K(x)\, A^+(x,t)\, dx + \alpha A_x^+(0,t) - \beta A^+(0,t). \tag{4.6}$$

Since $f^+ \in L^1(\mathbf{R}_+)$,

$$N^+(\lambda) = i\lambda\alpha e^{iw(0)} - \alpha[ip(0)\, e^{iw(0)} + A^+(0,0)] - \beta e^{iw(0)} + o(1),$$
$$\lambda \in \overline{\mathbf{C}}_+, \quad |\lambda| \to \infty \tag{4.7}$$

by (4.5). The result follows from (4.7). ▮

Now, let us assume that

$$\int_0^\infty e^{\varepsilon x}\{|q(x)| + |p(x)| + |p'(x)| + |K(x)|\}\, dx < \infty, \qquad \varepsilon > 0. \tag{4.8}$$

THEOREM 4.3. *Under the condition* (4.8) *the operator L has a finite number of eigenvalues and spectral singularities. Each of them is of finite multiplicity.*

We can prove Theorem 4.3 using analytic continuation. ([22], see also [4–6; 11; 17–19; 23]).

Now, let us discuss whether the hyphothesis of Theorem 4.3 can be weakened to attain the same result. For this we will assume that

$$\lim_{x\to\infty} p(x) = 0, \qquad \sup_{x\in\mathbf{R}_+} \{e^{\varepsilon\sqrt{x}}[|q(x)| + |p'(x)| + |K(x)|]\} < \infty, \quad \varepsilon > 0. \tag{4.9}$$

From (2.2), (2.4), and (2.5), condition (4.9) guarantees the function N^+ is analytic in $\mathbf{C}_+$, and all of its derivatives are continuous in $\overline{\mathbf{C}}_+$. So

$$|N^+(\lambda) - i\lambda\alpha e^{iw(0)}| < \infty, \qquad \lambda \in \overline{\mathbf{C}}_+, \tag{4.10}$$

and

$$\left|\frac{d^r}{d\lambda^r} N^+(\lambda)\right| \leqslant A_r, \qquad \lambda \in \overline{\mathbf{C}}_+, \quad r = 1, 2, ..., \tag{4.11}$$

where

$$A_r = 2^r c \int_0^\infty t^r \exp\left(-\frac{\varepsilon}{2}\sqrt{t}\right) dt, \qquad r = 1, 2, ..., \tag{4.12}$$

and $c > 0$ is a constant.

Let us denote the set of all limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+, respectively, and the set of all zeros N^+ with infinite multiplicity in $\overline{\mathbf{C}}_+$ by M_5^+.

Then

$$M_3^+ \subset M_2^+, \qquad M_4^+ \subset M_2^+, \qquad M_5^+ \subset M_2^+.$$

Using of the continuity of all derivatives of N^+ on the real axis we have

$$M_3^+ \subset M_5^+, \qquad M_4^+ \subset M_5^+. \tag{4.13}$$

We will use the following uniqueness theorem of Pavlov for the analytic functions on the upper half-plane, to prove the next result.

PAVLOV'S THEOREM. *Let us assume that the function f is analytic in $\mathbf{C}_+$, all of its derivatives are continuous up to the real axis, and there exist $T > 0$ such that*

$$|f^{(r)}(z)| \leqslant A_r, \qquad r = 0, 1, ..., \quad z \in \overline{\mathbf{C}}_+, \quad |z| < 2T, \tag{4.14}$$

and

$$\left| \int_{-\infty}^{-T} \frac{\ln |f(x)|}{1+x^2} \, dx \right| < \infty, \qquad \left| \int_{T}^{\infty} \frac{\ln |f(x)|}{1+x^2} \, dx \right| < \infty. \tag{4.15}$$

If the set Q, with linear Lebesque measure zero, is the set of all zeros of the function f with infinite multiplicity and if

$$\int_0^h \ln E(s) \, d\mu(Q_s) = -\infty,$$

where $E(s) = \inf_r A_r s^r / r!$, $r = 0, 1, 2, ...$, $\mu(Q_s)$ is the linear Lebesque measure of s-neighborhood of Q and h is an arbitrary positive constant ([24]), *then $f(z) \equiv 0$.*

LEMMA 4.4. $M_5^+ = \varnothing$.

Proof. It is easy to see from Lemma 4.2 and (4.10), (4.11) that N^+ satisfies (4.14) and (4.15). Since the function N^+ is not equal to zero identically, then by the Pavlov's Theorem, M_5^+ satisfies

$$\int_0^h \ln E(s) \, d\mu(M_{5,s}^+) > -\infty, \tag{4.16}$$

where $E(s) = \inf_r A_r s^r / r!$, $\mu(M_{5,s}^+)$ is the linear Lebesque measure of s-neighborhood of M_5^+, and the constant A_r is defined by (4.12).

Now we will obtain the following estimates for A_r,

$$A_r = 2^r c \int_0^\infty t^r \exp\left(-\frac{\varepsilon}{2}\sqrt{t}\right) dt \leqslant B b^r r^r r!, \tag{4.17}$$

where B and b are constants depending on c and ε. From (4.17) we find

$$E(s) = \inf_r \frac{A_r s^r}{r!} \leqslant B \inf_r \{b^r s^r r^r\} \leqslant B \exp\{-b^{-1} e^{-1} s^{-1}\},$$

or by (4.16)

$$\int_0^h \frac{1}{s}\, d\mu(M_{5,s}^+) < \infty. \tag{4.18}$$

(4.18) holds for an arbitrary s, if and only if $\mu(M_{5,s}^+) = 0$ or $M_5^+ = \varnothing$. ∎

THEOREM 4.5. *Under the condition* (4.9) *the operator* L *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the functions N^+ and N^- have a finite number of zeros with finite multiplicities in $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. We prove it only for N^+. The case N^- is similar.

From Lemma 4.4 and (4.13) we find that $M_3^+ = M_4^+ = \varnothing$. So the bounded sets M_1^+ and M_2^+ have no limit points (See Lemma 4.2), i.e., the function N^+ has only a finite number of zeros in $\overline{\mathbf{C}}_+$. Since $M_5^+ = \varnothing$ these zeros are of finite multiplicity. ∎

Now we will discuss the structure of M_5^+ under a condition weaker than (4.9).

THEOREM 4.6. *If*

$$\lim_{x \to \infty} p(x) = 0,$$
$$\sup_{x \in \mathbf{R}^+} \{\exp(\varepsilon x^\delta)[\,|q(x)| + |p'(x)| + |K(x)|\,]\} < \infty, \qquad \varepsilon > 0, \quad 0 < \delta < \tfrac{1}{2}, \tag{4.19}$$

then

$$\sum_n (\ell_n^+)^{(1-2\delta)/(1-\delta)} < \infty,$$

where $\{\ell_n^+\}$ *is the sequence of lengths of all finite complementary intervals of* M_5^+.

Proof. From (2.4)–(2.7) and (4.19) we obtain

$$\left|\frac{d^r}{d\lambda^r} N^+(\lambda)\right| \leqslant Bb^r r! r^{r(1-\delta)/\delta}, \qquad \lambda \in \overline{\mathbf{C}}_+, \tag{4.20}$$

where B and b are constants depending on ε and δ.

Let $G_\theta = G_\theta(\mathbf{C}_+)$, $(0<\theta<1)$, be the Gevrey class of analytic functions in $\mathbf{C}_+$ and Φ_θ be the system of all sets of uniqueness for G_θ, [8]. Hence it follows from (4.20) that

$$N^+ \in G_{\delta/(1-\delta)}, \qquad M_5^+ \notin \Phi_{\delta/(1-\delta)}.$$

Then by Carleson Theorem we obtain

$$\sum_n (\ell_n^+)^{1-(\delta/(1-\delta))} < \infty$$

([2], see also [8]). ∎

The above theorem shows that the eigenvalues of L may not be of finite multiplicity if condition (4.19) is satisfied.

Note. It is well known that, Carleson theorem gives the condition necessary for a set M_5^+ not belong to $\Phi_{\delta/(1-\delta)}$. But, we have to note that, if condition (4.19) holds, then it would be necessary to investigate the structure of the set M_5^+ using the Hruscev Theorem (Theorem 1, [8]).

5. PRINCIPAL FUNCTIONS

In this section we assume (4.9). Let $\lambda_1^+, ..., \lambda_j^+$ and $\lambda_1^-, ..., \lambda_k^-$ denote the zeros of the functions N^+ in $\mathbf{C}_+$ and N^- in $\mathbf{C}_-$ (which are the eigenvalues of L) with multiplicities $m_1^+, ..., m_j^+$ and $m_1^-, ..., m_k^-$, respectively. Similarly, let $\lambda_1, ..., \lambda_v$ and $\lambda_{v+1}, ..., \lambda_\ell$ be zeros of N^+ and N^- in $\mathbf{R}^*$ (which are the spectral singularities of L) with multiplicities $n_1, ..., n_v$ and $n_{v+1}, ..., n_\ell$, respectively. Then

$$\psi^+(x,\lambda_i^+), \left\{\frac{\partial}{\partial\lambda}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}, ..., \left\{\frac{\partial^{m_i^+-1}}{\partial\lambda^{m_i^+-1}}\psi^+(x,\lambda)\right\}_{\lambda=\lambda_i^+},$$

and

$$\psi^-(x, \lambda_i^-), \left\{\frac{\partial}{\partial\lambda}\psi^-(x, \lambda)\right\}_{\lambda=\lambda_i^-}, ..., \left\{\frac{\partial^{m_i^- -1}}{\partial\lambda^{m_i^- -1}}\psi^-(x, \lambda)\right\}_{\lambda=\lambda_i^-}$$

are called the principal functions corresponding to the eigenvalues $\lambda=\lambda_i^+$, $i=1, 2, ..., j$ and $\lambda=\lambda_i^-$, $i=1, 2, ..., k$ of L, respectively, where $\psi^{\pm}(x,\lambda)$ are defined by (4.1). Similarly

$$\psi(x, \lambda_i), \left\{\frac{\partial}{\partial\lambda}\psi(x, \lambda)\right\}_{\lambda=\lambda_i}, ..., \left\{\frac{\partial^{n_i-1}}{\partial\lambda^{n_i-1}}\psi(x, \lambda)\right\}_{\lambda=\lambda_i},$$
$$i=1, ..., v, v+1, ..., \ell$$

are the principal functions corresponding to the spectral singularities of L, where $\psi(x, \lambda)$ is defined by (4.2).

From (2.8.), (4.1) and (4.3) we obtain that the principal functions corresponding to the eigenvalues of L are in $L^2(\mathbf{R}_+)$.

Let us introduce the Hilbert spaces

$$H_+ = \left\{f: \int_0^\infty (1+x)^{2n_0}\,|f(x)|^2\,dx<\infty\right\},$$

$$H_- = \left\{g: \int_0^\infty (1+x)^{-2n_0}\,|g(x)|^2\,dx<\infty\right\}$$

with

$$\|f\|_+^2 = \int_0^\infty (1+x)^{2n_0}\,|f(x)|^2\,dx, \qquad \|g\|_-^2 = \int_0^\infty (1+x)^{-2n_0}\,|g(x)|^2\,dx,$$

respectively, where $n_0=\max\{n_1, ..., n_v, n_{v+1}, ..., n_\ell\}+1$. It is clear that

$$H_+ \subset_{\neq} L^2(\mathbf{R}_+) \subset_{\neq} H_-$$

H_- is isomorphic to the dual of H_+ ([1]).

THEOREM 5.1.

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi(\cdot, \lambda)\right\}_{\lambda=\lambda_i} \notin L^2(\mathbf{R}_+), \qquad n=0, 1, ..., n_i-1, \quad i=1, ..., v, v+1, ..., \ell$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi(\cdot, \lambda)\right\}_{\lambda=\lambda_i} \in H_-, \qquad n=0, 1, ..., n_i-1, \quad i=1, ..., v, v+1, ..., \ell.$$

Proof. Let $0 \leqslant n \leqslant n_i - 1$ and $1 \leqslant i \leqslant v$. Using (4.2.) we have

$$\left\{\frac{\partial^n}{\partial \lambda^n} \psi(x, \lambda)\right\}_{\lambda = \lambda_i} = \sum_{j=1}^{n} A_j(\lambda_i) \left\{\frac{\partial^j}{\partial \lambda^j} e^+(x, \lambda)\right\}_{\lambda = \lambda_i}, \tag{5.1}$$

where

$$A_j(\lambda_i) = -\binom{n}{j}\left\{\frac{\partial^{n-j}}{\partial \lambda^{n-j}} N^-(\lambda)\right\}_{\lambda = \lambda_i}.$$

The proof of theorem is obtained from (2.2), (2.8), and (5.1.) In a similar way we may also prove the result for $0 \leqslant n \leqslant n_i - 1$ and $v + 1 \leqslant i \leqslant \ell$. ∎

6. SPECIAL CASES OF L

a. *Sturm–Liouville Operator* L_0

In this case the condition (4.9) assumes the form of

$$\sup_{x \in \mathbf{R}_+} \{e^{\varepsilon \sqrt{x}} |q(x)|\} < \infty, \qquad \varepsilon > 0. \tag{6.1}$$

Hence, from Theorem 4.5 we find

Corollary 6.1. *Under the condition* (6.1) *the operator* L_0 *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Note that the same result has been obtained by Naimark [22] under the stronger assumption

$$\int_0^\infty e^{\varepsilon x} |q(x)|\, dx < \infty, \qquad \varepsilon > 0$$

(see also [17], [23]).

b. *Sturm–Liouville Operator* L_1 *with Integral Boundary Condition*

Here the condition (4.9) assumes the form of

$$\sup_{x \in \mathbf{R}^+} \{e^{\varepsilon \sqrt{x}} [\, |q(x)| + |K(x)|\,]\} < \infty, \tag{6.2}$$

for the operator L_1. From Theorem 4.5 we deduce the following.

COROLLARY 6.2. *If the condition* (6.2) *holds, the operator* L_1 *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

c. *The Quadratic Pencil of the Schrödinger Operator* L_2.

In this case the condition (4.9) will assume the form

$$\lim_{x\to\infty} p(x)=0, \qquad \sup_{x\in\mathbf{R}^+} \{e^{\varepsilon\sqrt{x}}[\,|q(x)|+|p'(x)|\,]\}<\infty, \quad \varepsilon>0. \tag{6.3}$$

Similarly, from Theorem 4.5, we have

COROLLARY 6.3. *Under the condition* (6.3) *the operator* L_2 *has a finite number of eigenvalues and spectral singularities. Each of them is of finite multiplicity.*

The same result has been proved by Maksudov [18] under the stronger assumption

$$\int_0^\infty e^{\varepsilon x}\{\,|q(x)|+|p(x)|+|p'(x)|\}\,dx<\infty, \qquad \varepsilon>0,$$

and by Jaulent–Jean [10] under the assumption

$$\sigma_{ss}(L_2)=\varnothing.$$

d. *Klein–Gordon s-Wave Operator*

In this case we obtain

COROLLARY 6.4. *If*

$$\lim_{x\to\infty} p(x)=0, \qquad \sup_{x\in\mathbf{R}^+} \{e^{\varepsilon\sqrt{x}}\,|p'(x)|\}<\infty, \qquad \varepsilon>0,$$

then the operator L_3 *has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Under Condition (4.9), the eigenfunction expansion in terms of the principal functions of L will consist of the subject of another article.

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REFERENCES

1. Yu. M. Berenzanski, "Expansion in Eigenfunctions of Selfadjoint Operators," Amer. Math. Soc., Providence, 1968.
2. L. Carleson, Sets of uniqueness for functions regular in the unit circle, *Acta Math.* **87** (1952), 325–345.
3. A. Degasperis, On the inverse problem for the Klein–Gordon *s*-wave equation, *Jour. Math. Phys.* **11**(2) (1970), 551–567.
4. M. G. Gasymov, On the decomposition in a series of eigenfunctions for a non-selfadjoint boundary value problem of the solution of a differential equation with a singularity at a zero point, *Soviet Math. Dokl.* **6** (1965), 1426–1429.
5. M. G. Gasymov, Expansion in terms of the solutions of a scattering theory problem for the non-selfadjoint Schrödinger equation, *Soviet Math. Dokl.* **9** (1968), 390–393.
6. M. G. Gasymov and F. G. Maksudov, The principal part of the resolvent of non-selfadjoint operators in neighbourhood of spectral singularities, *Funct. Anal. Appl.* **6** (1972), 185–192.
7. W. Greiner, "Relativistic Quantum Mechanics, Wave Equations," Springer-Verlag, Berlin/New York, 1994.
8. S. V. Hruscev, Sets of Uniqueness for the Gevrey classes, *Ark. Math.* **15**(2) (1977), 253–304.
9. S. V. Hruscev, Spectral singularities of dissipative schrödinger operator with rapidly decreasing potential, *Indiana Univ. Math. J.* **33** (1984), 313–338.
10. M. Jaulent and C. Jean, The inverse *s*-wave scattering problem for a class of potentials depending on energy, *Comm. Math. Phys.* **28** (1972), 177–220.
11. R. R. D. Kemp, A singular boundary value problem for a non-selfadjoint differential operator, *Canad. J. Math.* **10** (1958), 447–462.
12. A. M. Krall, The adjoint of a differential operator with integral boundary conditions, *Proc. Amer. Math. Soc.* **16** (1965), 738–742.
13. A. M. Krall, A nonhomogeneous eigenfunction expansion, *Trans. Amer. Math. Soc.* **117** (1965), 352–361.
14. A. M. Krall, Second order ordinary differential operators with general boundary conditions, *Duke J. Math.* **32** (1965), 617–625.
15. A. M. Krall, Nonhomogeneous differential operators, *Michigan Math. J.* **12** (1965), 247–255.
16. A. M. Krall, On non-selfadjoint ordinary differential operators of second order, *Sov. Math. Dokl.* **165** (1965), 1235–1237.
17. V. E. Lyance, "A Differential Operator with Spectral Singularities, I, II," Amer. Math. Soc. Trans. Ser. 2, Vol. 60 (1967), 185–225, 227–283.
18. F. G. Maksudov, "Eigenfunction Expansion of a Quadratic Pencil of Schrödinger Operators, Spectral Theory of Operators," Bakü, 1977. [In Russian]
19. F. G. Maksudov and B. P. Allakhverdiev, Spectral analysis of a new class of non-selfadjoint operators with continuous and point spectrum, *Sov. Math. Dokl.* **30** (1984), 566–569.
20. F. G. Maksudov and G. Sh. Guseinov, On solution of the inverse scattering problem for a Quadratic pencil of one-dimensional Schrödinger operators on the whole axis, *Sov. Math. Dokl.* **34** (1987), 34–38.
21. V. A. Marchenko, "Expansion in Eigenfuctions of Nonselfadfoint Singular Second Order Differential Operators," Amer. Math. Soc. Trans. Ser. 2, Vol. 25, pp. 77–130, Amer. Math. Soc., Providence, 1963.
22. M. A. Naimark, "Investigation of the Spectrum and the Expansion in Eigenfunctions of a Non-selfadjoint Operator of Second Order on a Semi-axis," Amer. Math. Soc. Trans. Ser. 2, Vol. 16, pp. 103–193, Amer. Math. Soc., Providence, 1960.

23. M. A. Naimark, “Linear Differential Operators I, II,” Ungar, New York, 1968.
24. B. S. Pavlov, The non-selfadjoint Schrödinger operator, *Topics in Math. Phys.* **1** (1967), 87–110.
25. B. S. Pavlov, On separation conditions for the spectral components of a dissipative operators, *Math. USSR Izvest*. **9** (1975), 113–137.
26. J. T. Schwartz, Some non-selfadjoint operators, *Comm. Pure and Appl. Math.* **13** (1960), 609–639.
27. E. C. Titchmarsh, “Eigenfunction Expansions Associated with Second Order Differential Equations,” Oxford Univ. Press, London, 1962.